

# Winding Strings in Singular Spacetimes

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## ABSTRACT:

Evolution of winding strings in spacetimes with cycles whose proper lengths depend on time is examined. It was established earlier that extended objects wrapping the shrinking dimension in compactified Milne spacetime enjoy classically nonsingular evolution. Extensions of this observation to other spacetimes are discussed.

**KEYWORDS:** String theory, Cosmology, Big Bang, Compactified Milne Spacetime.

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## 1. Introduction

One of the most fascinating questions in cosmology concerns the nature of the apparent singularity assumed to lie at the origin of the Big Bang. For many reasons this question cannot be resolved within classical general relativity, and it is natural to seek insights based on string theory, which has provided numerous examples of consistent physics even in background spacetimes which are classically singular. Perturbative examples of this include, for example, Euclidean orbifold compactifications [1]. This has fueled hopes that the initial cosmological singularity would also find a consistent description, perhaps involving a smooth transition from what has been termed a “pre-Big-Bang” era [2]. In recent years a lot of attention was devoted to studies of string theory spacetimes with cosmologically relevant features such as time-dependence and occurrence of space-like singularities. These studies focused mostly on various time-dependent orbifolds [3]-[11] (the subject is reviewed

in references [12] and [13]). This activity was hampered by the lack of a proper second-quantized description of string theory, but some semi-heuristic approaches to this have been formulated [14].

One of the simplest examples of a spacetime background with a space-like singularity is compactified Milne space [4]. Interest in this particular case is motivated by its apparent simplicity, as well as its possible significance for the cyclic Universe scenario [15, 16] (as realized in the heterotic M-theory framework [17, 18]). This spacetime involves a compact direction whose radius decreases linearly with time  $t$  until it reaches zero, and then expands linearly at the same rate. Point particles, as well as generic configurations of extended objects, experience blue-shifting as the radius shrinks. This effect disappears if the momentum in the shrinking direction vanishes, but even for such special initial conditions the problem cannot be dismissed, since any infinitesimal momentum in the shrinking direction will lead to nonanalytic behavior of generic metric perturbations [19] as  $t$  goes to zero. Turok et al. [19] argue that for those states one has to seek a description based on an expansion in inverse powers of  $\alpha'$ , i.e. opposite to the expansion which leads to Einstein's gravity at large distances. One may adopt also a somewhat simple-minded view according to which the blue-shifted objects decouple leaving an effective theory valid close to the singularity which now involves only states corresponding to objects whose mass remains finite in that region. The authors of [19] point out however that in contrast to point particles, states of extended objects winding uniformly around the shrinking direction evolve classically in a smooth and unambiguous way through the singularity. This is due to the fact that for a wrapped object the dimension which is wrapped ceases to be a direction of possible physical motion. The main focus of [19] is actually on winding states of M2-branes in M-theory, but the case of winding strings is interesting in its own right. In the M-theory context such states arise as M2-branes wrapping tori, whose moduli are time-dependent. Zero-modes of strings wrapped on the Milne circle can be viewed either as particles with time-dependent mass in Minkowski space, or as particles of fixed mass in a spacetime geometry which is singular due to a conformal factor which vanishes at  $t = 0$ . Unambiguous and smooth evolution of particles in such a situation is somewhat surprising, and one would like

to understand better how this comes about. A part of section 2 below is devoted to this issue.

This note extends the considerations of winding strings and membranes appearing in [19] to somewhat more general background spacetimes involving shrinking cycles. Such backgrounds are of interest to cosmology, since there is no reason to assume that only one compact dimension undergoes periodic expansion and contraction, as in the simplest incarnations of the cyclic universe scenario. Indeed, such spacetimes have recently been the focus of much activity aimed at understanding the current phase of accelerated expansion of the universe [20]-[24]. It appears however that the observation of Turok et al. does not directly generalize to cases when more than one cycle is shrinking. The reasons for this are discussed in section 5.

The considerations reported below are presented in terms of bosonic string theory. More realistically one would need to embed this in a superstring setting, which involves additional degrees of freedom on the worldsheet. In particular, choices need to be made as to the boundary conditions for spacetime fermions when they are continued around any cycles. The mass and interpretation of quantum string states corresponding to the classical winding modes discussed here will depend on how this is done. This note does not address this issue; the considerations presented here are classical, in the spirit of [19]. It should also be kept in mind that considerations of time-dependent backgrounds of this type require verifying consistency at the quantum level (which for strings implies checking beta-functions for the worldsheet sigma model). Examples are known where this can be done, but this aspect of the problem is left for future work.

The focus on winding modes in a cosmological setting has been a leitmotif in string cosmology since the early papers by Brandenberger and Vafa [25]. Quite recently the condensation of light winding states in a context very close to the one discussed here appeared in considerations of chronology protection [26].

While this note was being written a couple of papers [27, 28] appeared, where similar issues are addressed.

## 2. Strings in Milne Spacetime

This section is devoted to the example of compactified Milne space discussed in [19]. Compactified Milne spacetime has the metric

$$ds^2 = -dt^2 + \beta^2 t^2 d\theta^2 + dx^k dx^k, \quad k = 1 \dots d-1, \quad (2.1)$$

where the coordinate  $\theta$  describes an  $S^1$ , so that

$$\theta \approx \theta + 2\pi, \quad (2.2)$$

and  $\beta t$  is the (instantaneous) radius<sup>1</sup> of the  $S^1$ . The coordinate  $t$  is the time, assumed to run from  $-\infty$  to  $\infty$ . This metric is locally flat, but it has a curvature singularity at  $t = 0$  unless  $\beta = 1$ . In the latter case the metric (2.1) can be brought to flat Minkowski form (globally) by a change of coordinates.

The classical dynamics of a relativistic string in a background metric  $G$  can be described by means of the Nambu-Goto action [31, 32]

$$S = -\frac{1}{4\pi\alpha'} \int d\tau d\sigma \sqrt{(\partial_\tau X \cdot \partial_\sigma X)^2 - (\partial_\tau X \cdot \partial_\tau X)(\partial_\sigma X \cdot \partial_\sigma X)}, \quad (2.3)$$

where  $X^\mu(\tau, \sigma)$  denote the string embedding and the “dot” denotes the scalar product defined by  $G$ . Using the notation  $(X^0, \dots, X^d) \equiv (T, X^1, \dots, X^{d-1}, \Theta)$  the string zero-modes in the winding sector labeled by  $w$  are of the form

$$T = t(\tau), \quad X^k = x^k(\tau), \quad \Theta = w\sigma. \quad (2.4)$$

The reason why these modes are of special interest was forcefully put forth in [19]. Objects moving in the spacetime (2.1) will undergo blue-shifting as soon as they have any momentum along the shrinking direction. Their description becomes invalid as the singularity is approached. The exception are extended objects wrapped around the shrinking dimension. For such objects motion in this direction is a gauge degree of freedom, so these objects are naturally immune to blue-shifting. Indeed, their mass is to first approximation given by their tension times their length, and since

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<sup>1</sup>In fact, as discussed in [19], one could just as well have a segment instead of  $S^1$  here. The only important point is that this dimension is shrinking as the singularity is approached.

their length tends to zero as the classical singularity is approached, they become light (and instantaneously massless) as it is crossed.

The action in the zero-mode sector follows from the Nambu-Goto action upon substituting the metric (2.1) and the ansatz (2.4). It is given by

$$S_0 = - \int d\tau \, m(t) \sqrt{\dot{t}^2 - \dot{\vec{x}}^2} , \quad (2.5)$$

where  $\vec{x}$  denotes the transverse coordinates  $x^1 \dots x^{d-1}$ , the dot denotes a derivative with respect to  $\tau$ , and

$$m(t) = \frac{|wt|\beta}{2\alpha'} \equiv m_0 |t| . \quad (2.6)$$

This can be interpreted as the action for a particle in Minkowski space, with a mass depending on time according to (2.6).

It is convenient to fix the gauge  $T = \tau$  (synchronous gauge), which leaves the transverse coordinates  $x^k$  as the independent degrees of freedom. The action becomes

$$S_0 = - \int dt \, m(t) \sqrt{1 - \dot{\vec{x}}^2} . \quad (2.7)$$

The momentum conjugate to  $\vec{x}$

$$\vec{p} = \frac{m(t)\dot{\vec{x}}}{\sqrt{1 - \dot{\vec{x}}^2}} \quad (2.8)$$

is conserved. The Hamiltonian is simply

$$H = \sqrt{\vec{p}^2 + m^2(t)} . \quad (2.9)$$

As stressed by Turok et al. [19] this is not singular at  $t = 0$ . The simple-minded argument given above has also been carried through in a more formal manner (in the Hamiltonian formalism) leading to the same conclusions [19]. The equations of motion

$$\frac{d\vec{x}}{dt} = \frac{\vec{p}}{\sqrt{\vec{p}^2 + m(t)^2}} \quad (2.10)$$

are solved uniquely by

$$\vec{x} = \vec{x}_0 + \frac{\vec{p}}{m_0} \operatorname{asinh}\left(\frac{m_0}{|\vec{p}|} t\right) , \quad (2.11)$$

where  $\vec{x}_0$  is an integration constant [19].

One can also find this solution starting with the Polyakov form of the action in conformal gauge [31, 32]:

$$S = -\frac{1}{4\pi\alpha'} \int d\sigma d\tau (-\partial_+ T \partial_- T + T^2 (\partial_+ \Theta \partial_- \Theta) + \partial_+ \vec{X} \partial_- \vec{X}) \quad (2.12)$$

and imposing Virasoro constraints  $T_{++} = T_{--} = 0$ , where

$$T_{\pm\pm} = -\partial_{\pm} T \partial_{\pm} T + T^2 (\partial_{\pm} \Theta \partial_{\pm} \Theta) + \partial_{\pm} \vec{X} \partial_{\pm} \vec{X} \quad (2.13)$$

are the nontrivial components of the worldsheet energy-momentum tensor in light-front coordinates  $\sigma^{\pm} \equiv \tau \pm \sigma$ .

There is also an alternative way of interpreting the action (2.5): one can regard it as the action for a particle with fixed mass in a spacetime with metric

$$ds^2 = t^2 (-dt^2 + dx^k dx^k) . \quad (2.14)$$

This metric is singular: its determinant vanishes at  $t = 0$ . Invariants built from the Riemann tensor also diverge there, so from the point of view of differential geometry this is a real singularity, rather than a coordinate one.

It is illuminating to write the metric (2.14) in terms of a different time coordinate, namely  $|\xi| = t^2/2$ :

$$ds^2 = -d\xi^2 + 2|\xi| dx^k dx^k . \quad (2.15)$$

If  $\xi$  is positive this is the Friedman-Robertson-Walker metric for a universe filled with radiation and  $\xi$  would be identified with the cosmic time ( $t$  being the conformal time). Allowing negative cosmic times in (2.15) is an extension of the standard Friedman-Robertson-Walker spacetime, but in contrast to the usual sense of extending the domain of validity of the coordinate system, it does not remove singularity at  $\xi = 0$  ( $t = 0$ ) which is a real singularity of the curvature. In spite of this fact, one is allowed to consider time like geodesics describing smooth evolution of massive particles through the singular locus in the extended Friedman-Robertson-Walker spacetime. To understand this point one has to consider the interpretation of the  $\xi = 0$  locus more carefully. The metric becomes singular as  $\xi$  goes to zero. In the standard Friedman-Robertson-Walker models the singularity in the past is generally

interpreted as a signal that space-time has originated from a point<sup>2</sup>. In fact, as long as one does not need to assume that the set  $\xi = 0$  has some intrinsic structure (even only a dimension in a topological sense) any assumption about such an intrinsic structure could be rejected. But the apparent lack of need is not a sufficient argument to definitely exclude such possibility. The alternatives could be studied as well as their possible physical consequences. One may argue that while the metric has a real singularity, the problem of finding a time-like curve of minimum metric length is consistent in spite of this fact because the length of any time-like curve is perfectly well defined even if it happens to cross the singular locus. Therefore, assuming that at  $\xi = 0$  there is a null three-dimensional surface rather than a point, the variational problem is well posed and as has a unique solution (2.11). Within the framework of general relativity one cannot determine how the singular locus is to be interpreted. Only physical consequences – if found and confirmed – could justify *a posteriori* or reject the above assumption. But from the string perspective the situation is clear: the interpretation of the coordinates  $x$  is given – they are global Minkowski coordinates and the issue of degeneration does not arise. At  $t = 0$  only the compact dimension shrinks to a point. There is no singularity of the  $d$ -dimensional metric<sup>3</sup>. At  $t = 0$  there is a  $d - 1$ -dimensional plane with Minkowski metric (locally on the plane and its neighborhood). The particle states identified with fundamental modes of winding strings become massless as  $t$  approaches zero. Particle masses vary according to the time-dependent conformal factor. Particle dynamics is formally equivalent to dynamics of constant mass particles following geodesics in the conformally flat space-time.

Is this a unique, somewhat pathological example? Its stringy origin clearly indicates that it is not. In the following section this question is taken up in a class of spacetimes generalizing (2.1).

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<sup>2</sup>The locus  $\xi = 0$  is formally spanned by a three-dimensional hyperspace in the space of coordinates, but the singularity of the metric at  $\xi = 0$  implies that the lengths of all formal curves on this hyperspace are equal to zero. It could then be argued that this null hyperspace physically is just a point.

<sup>3</sup>The original Milne metric (2.1) is  $d + 1$ -dimensional.



### 3. Winding states in general cases with a shrinking cycle

This section is devoted to moving beyond the case of compactified Milne spacetime. It is rather clear that the example reviewed in the previous section is easy to generalize. A similar picture should be valid whenever there is a cycle which degenerates with time. Strings wrapped on it will have zero-modes which to an observer probing distances much larger than the radius of the cycle will appear as particles with a time-dependent mass. This argument should be valid at least in an appropriate adiabatic limit.

To be more explicit, let us focus on metrics which admit an isometry. By a choice of coordinates one can always express such a metric as

$$ds^2 = e^{2\phi}(du + A)^2 + ds_\perp^2, \quad (3.1)$$

where

$$ds_\perp^2 = h_{\mu\nu}dx^\mu dx^\nu, \quad A = A_\mu dx^\mu \quad (3.2)$$

and  $x^\mu, u$  are the spacetime coordinates with  $\mu = 0 \dots d-1$ . It will be assumed that  $\phi, h_{\mu\nu}, A_\mu$  do not depend on the coordinate  $u$ . This admits the interpretation of  $u$  as a coordinate on an  $S^1$ , that is, subject to periodic identification.

Let us consider zero-modes of strings in this background. The zero-mode ansatz reads:

$$(X^A(\sigma, \tau)) = (t(\tau), x^1(\tau), \dots, x^{d-1}(\tau), w\sigma). \quad (3.3)$$

This class of configurations describes ground states of a string winding  $w$  times around the direction  $u \equiv x^d$ . The Nambu-Goto action for these modes reduces to

$$S_0 = - \int d\tau \, m \sqrt{-h_{\mu\nu} \dot{x}^\mu \dot{x}^\nu}, \quad (3.4)$$

where now

$$m = \frac{|w|}{2\alpha'} e^\phi. \quad (3.5)$$

This has an obvious interpretation. From the point of view of winding string zero-modes only the spacetime transverse to the cycle is visible. The Nambu-Goto action reduces to that of a particle with mass given by (3.5) moving in the geometry described by the metric  $ds_\perp^2$ . This mass is just the string tension multiplied by the

proper length of the winding string. In the spirit of [19] one can make this argument more formally in the Hamiltonian formalism – this is sketched in the appendix.

Suppose now that  $\phi$  depends only on time and that  $h_{\mu\nu}$  is static. In that case, at least for slowly varying  $\phi$ , it is natural to regard  $R(t) \equiv e^\phi$  as the radius of the compact direction. If the original spacetime involves a cycle which shrinks to zero and re-expands (for example, if  $\phi \sim \gamma \ln(|t|)$  for positive  $\gamma$ ), then the situation is similar to the Milne example.

Equivalently, one can regard (3.4) as the action of a massive particle in a space-time described by the metric

$$ds^2 = e^{2\phi} h_{\mu\nu} dx^\mu dx^\nu . \quad (3.6)$$

These two possibilities clearly correspond to the freedom of choosing either the Einstein frame or the string frame in the field theory low energy effective action. The metric (3.6) has a real singularity under the above assumptions, but (as discussed in the previous section) one expects that there should be smooth geodesics passing through the singular locus. While it may be hard to find geodesics for a specific metric of the form (3.6) explicitly, in many cases one may confirm this expectation by computing the Hamiltonian and verifying that it is not singular.

## 4. Examples

This section is devoted to some simple examples. First consider spacetimes of the form

$$ds^2 = -dt^2 + \beta^2 t^{2n} d\theta^2 + d\vec{x}^2 , \quad (4.1)$$

for integers  $n > 1$ . In this case there is a curvature singularity regardless of the periodicity of  $\theta$ . The discussion of winding strings can be carried out just as for the Milne case discussed in section (2), the only modification being that the effective mass (2.6) has to be replaced by

$$m(t) = \frac{|w|\beta}{2\alpha'} |t|^n . \quad (4.2)$$

The equation of motion (2.10) can also be integrated analytically, giving a smooth and unique answer which can be expressed in terms of elliptic functions. It is amusing

to note that defining the “cosmic time”  $\xi = t^n$  one has a Friedman-Robertson-Walker spacetime with  $a(\xi) = ((n+1)\xi)^{\frac{n}{n+1}}$  which corresponds to the equation of state  $p = \kappa\rho$  with  $\kappa = (2 - n)/3n$ . Thus the case  $n = 2$  corresponds to pressureless dust.

Another simple example is the metric considered by a number of authors (see e.g. [29]) as an analytic continuation of Witten’s two-dimensional black hole [30]:

$$ds^2 = -dt^2 + \beta^2 \tanh^2 t \, d\theta^2 + d\vec{x}^2 . \quad (4.3)$$

The metric (4.3) describes a curved spacetime: unlike the locally flat Milne example, in this case the Riemann tensor is nontrivial. Similarly to the Milne case the locus  $t = 0$  is not singular unless  $\theta$  is compactified: one finds, e.g. that

$$R_{\mu\nu\lambda\rho} R^{\mu\nu\lambda\rho} = 10 \operatorname{sech}^4 t , \quad (4.4)$$

which vanishes at  $t = 0$ . However, when the coordinate  $\theta$  is periodically identified so as to describe a compact dimension, there is a conical singularity similar to what occurs for the Milne spacetime. The Nambu-Goto action for zero-modes of the form (3.3) in synchronous gauge reduces to the same for as (2.7) but with

$$m(t) = \frac{\beta}{2\alpha'} |w \tanh t| . \quad (4.5)$$

In this case it is also simple to integrate the equation of motion (2.10) analytically, and one finds a unique and nonsingular result. This is perhaps not surprising, since the metric (4.3) reduces to Milne at small  $t$ .

## 5. Multiple cycles

It is natural to ask whether the arguments given in the previous sections generalize to situations where there are multiple shrinking cycles. On physical grounds one expects that any non-vanishing momentum in those directions will lead to divergences as the cycles shrink. Thus, even if an extended objects is wrapping one shrinking cycle it will still have singular evolution if it can move in another direction which is shrinking. At the very least, one can say that the system will be unstable. It might seem at first glance that if all shrinking directions were wrapped, then blue-shift singularities

would not appear, by a directly generalizing the arguments of Turok et al. [19]. Whether this can work or not depends on whether there is enough gauge symmetry to render motion the wrapped dimensions unphysical (“pure gauge”). In the case of wrapped strings it is easy to see that already in the case of two cycles shrinking at the same time there is a linear combination of them which is “pure gauge” in the above sense, and an orthogonal direction which is not. Thus the system is unstable against blue-shift divergences once there is any momentum in the latter dimension. So for strings one can have zero-modes smoothly crossing the singular locus only in the case of a single shrinking dimension.

One can verify this explicitly in a simple class of spacetimes generalizing (3.1), namely

$$ds^2 = \sum_{i=1}^p e^{2\phi_i} (du_i + A^{(i)})^2 + ds_{\perp}^2 , \quad (5.1)$$

where one has  $p$  commuting isometries along the directions parameterized by  $u_i$  and

$$ds_{\perp}^2 = h_{\mu\nu} dx^{\mu} dx^{\nu}, \quad A^{(i)} = A_{\mu}^{(i)} dx^{\mu} . \quad (5.2)$$

Here  $\phi_i, h_{\mu\nu}, A_{\mu}^{(i)}$  depend only on the “transverse” coordinates  $x_{\perp} \equiv (x^1, \dots, x^{d-p})$  and possibly on  $t$ . Thus one can interpret the  $u^i$  as being subject to periodic identification<sup>4</sup>. For simplicity, suppose also that the “dilaton”  $\phi_i$  depend only on time. A specific example would be a string winding a two-dimensional torus with time-dependent radii:

$$ds^2 = -dt^2 + t^2(\beta_1^2(dx^d)^2 + \beta_2^2(dx^{d-1})^2) + ds_{\perp}^2 . \quad (5.3)$$

This is a simple generalization of the Milne example where two torus cycles shrink to zero radius and then expand at the same rate.

Suppose  $p$  cycles are wrapped by the string. In such a situation the zero-mode ansatz generalizing (3.3) takes the form

$$(X^A(\sigma, \tau)) = (t(\tau), x^1(\tau), \dots, x^{d-p}(\tau), w_1\sigma, \dots, w_p\sigma) , \quad (5.4)$$

where  $w_1 \dots w_p$  are winding numbers. This class of configurations describes ground states of a string winding around the isometry directions. The Nambu-Goto action

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<sup>4</sup>All periodic coordinates are taken to have a fixed periodicity  $2\pi$ .

(2.3) for these modes reduces to

$$S_0 = - \int d\tau \, m \sqrt{-h_{\mu\nu} \dot{x}^\mu \dot{x}^\nu} \, , \quad (5.5)$$

where now

$$m = \frac{1}{2\alpha'} \sqrt{\sum_i w_i^2 e^{2\phi_i}} \, . \quad (5.6)$$

The interpretation is that winding strings move in the transverse space and their mass is determined by their proper length. The Hamiltonian is given by (2.9) with the time-dependent mass given by (5.6). It would seem at this point that there is no blue-shift divergence. The problem with this reasoning is that already the Ansatz (5.4) does not allow for any motion in the isometry directions. If one allows for any nonvanishing momentum in these directions the blue-shift divergence will appear. There is no argument generalizing that of [19] that can prevent this from happening. The problem is not apparent, because in formula (5.4) there are no  $\tau$  dependent terms in the isometry directions. For the case of a single shrinking dimension this was justified by the analysis performed in [19], which pointed out that for strings uniformly winding this dimension motion in this direction is not physical. This no longer suffices if there is more than one shrinking cycle. This can be verified by analysing the equations of motion which follow from the Nambu-Goto action. Equivalently, one can analyse them starting with the Polyakov action (2.12) and the Virasoro constraints. It is easy to see that there is always only one linear combination of momenta which is set to zero without instability. In the Polyakov approach this direction is given by the linear combination of cycles defined by zero-modes of the Virasoro constraints<sup>5</sup>. The directions orthogonal to this dimension do however suffer from a blue-shift instability. A systematic analysis could also be carried out in the Hamiltonian formalism, as done by Turok et al. [19] for the case of Milne space. Clearly, for the case of higher  $p$ -branes there will also be limits to how many cycles can shrink without necessarily leading to instability.

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<sup>5</sup>Specifically, this linear combination can be read off from the level-matching constraint  $\bar{L}_0 = L_0$  (using standard notation).

## 6. Conclusions

It was noted in [19] that at the classical level extended objects winding the shrinking circle in Milne spacetime evolve smoothly through the singularity. This note discussed the extension of the arguments given there to some more general examples.

The main focus of [19] was the physics of winding membrane zero-modes. The discussion of M-theory membranes in Milne spacetime presented there generalizes to some of the more general spacetimes discussed above (i.e. (3.1) and (5.1)) in the case of winding strings. However when more than one cycle shrinks there are limits on the dimensionality of extended objects which evolve smoothly and are stable. Specifically, in case of winding strings such stable and smooth evolution (along the lines of [19]) is possible only if no more than one cycle is shrinking.

The winding string examples discussed earlier in this paper can be viewed as membranes multiply wrapped on tori with some of the torus cycles degenerating in time. Instead of the Nambu-Goto action for the string one begins with the membrane action and an appropriate zero-mode ansatz. This leads to a nonsingular Hamiltonian if all shrinking dimensions are wrapped (up to the limitations discussed in section 5). The effective description thus obtained pertains to a string if only one membrane dimension is wrapped, or a particle if two membrane dimensions are wrapped.

The arguments presented here were based on two approximations: they were limited to the classical approximation, and furthermore to a minisuperspace approximation which ignored all non-zero string modes. To draw firm conclusions about the physics of this problem it would be crucial to understand the validity of this procedure. This would require a proper quantum treatment, taking into account all worldsheet degrees of freedom, not just the embedding coordinates  $X$ . It is clearly fascinating to pursue these questions and understand what message can be inferred from the apparently smooth passage enjoyed by winding extended objects in singular spacetimes.

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## A. Appendix: Hamiltonian reduction of winding string dynamics

This section presents a more formal argument showing that the metrics (3.1) lead to nonsingular evolution. This is a simple generalization of the analysis carried out by Turok et al. [19]. The starting point is the Polyakov action which has the form

$$S = \int d\tau L(\tau), \quad L(\tau) := -\frac{\mu_1}{2} \int d\sigma \sqrt{-\gamma} \gamma^{ab} \partial_a X^A \partial_b X^B g_{AB}, \quad (\text{A.1})$$

where  $(X^A) \equiv (X^0, X^1, \dots, X^{d-1}, \Theta) \equiv (X^0, X^k, \Theta) \equiv (X^\mu, \Theta)$  are embedding functions of a string in spacetime with metric  $g_{AB}$ ,  $\gamma_{ab}$  is the intrinsic metric of the string worldsheet ( $\gamma$  is its determinant), and  $(a) \equiv (\tau, \sigma)$ .

Owing to reparameterization invariance of the action (A.1) the system has constraints. Detailed analysis of the corresponding constrained dynamics has been carried out by Turok et al. The final form of the corresponding Hamiltonian reads

$$H = \int d\sigma \left( \frac{1}{2} \tilde{A} \tilde{C} + \tilde{A}^1 \tilde{C}_1 \right), \quad (\text{A.2})$$

where  $\tilde{A}$  and  $\tilde{A}^1$  are arbitrary functions of  $\tau$  and  $\sigma$ , and where

$$\tilde{C} := \Pi_A \Pi_B g^{AB} + \mu_1^2 \dot{X}^A \dot{X}^B g_{AB} = 0, \quad \tilde{C}_1 := \Pi_A \dot{X}^A = 0, \quad (\text{A.3})$$

are first-class constraints ( $\Pi_A := \partial L / \partial \dot{X}^A$ ,  $\dot{X}^A := \partial X^A / \partial \tau$ ,  $\dot{X}^A := \partial X^A / \partial \sigma$ ).

In case of a string twisted around  $u$ -direction defined by (3.3) and in the gauge  $\tilde{A}^1 = 0$ , Hamilton's equations lead to the new constraint  $\Pi_U = 0$ . The set of constraints for  $\tilde{C}$ ,  $U - w\sigma$  and  $\Pi_U$  is not first-class because the Poisson bracket between  $\tilde{C}$  and each of the other two constraints does not vanish. The solution is treating  $U = w\sigma$  and  $\Pi_U = 0$  as the second-class constraints. In that scheme the Dirac bracket amounts to canceling the  $U$  and  $\Pi_U$  derivatives from the Poisson bracket. This way one can eliminate the variables  $U$  and  $\Pi_U$  from the dynamics. As the result the dynamics of a string in  $d + 1$  dimensional spacetime reduces to the dynamics of a 'particle' in spacetime with the dimension  $d$ . The corresponding Hamiltonian can be obtained by substituting  $U = w\sigma$  and  $\Pi_U = 0$  into (A.3). Finally, one gets

$$H = \tilde{A}(\tau) \tilde{C} = \tilde{A}(\tau) (\Pi_\mu \Pi_\nu \tilde{h}^{\mu\nu} + \mu_1^2 w^2 e^{2\phi}), \quad (\text{A.4})$$

where  $\tilde{h}^{\mu\nu}$  is the inverse of the metric  $\tilde{h}_{\mu\nu} = h_{\mu\nu} + A_\mu A_\nu e^{2\phi}$  defined by (3.1).

It results from (A.4) that a string twisted around the direction  $u$ , behaves like a particle with an effective mass  $\mu_1 w e^\phi$  in spacetime with metric  $\tilde{h}_{\mu\nu}$ . The character of dynamics defined by the Hamiltonian (A.4) depends on regular/singular properties of  $\tilde{h}_{\mu\nu}$ .



## References

- [1] L. J. Dixon, J. A. Harvey, C. Vafa and E. Witten, “Strings On Orbifolds,” Nucl. Phys. B **261**, 678 (1985).
- [2] M. Gasperini and G. Veneziano, “The pre-big bang scenario in string cosmology,” Phys. Rept. **373**, 1 (2003) [arXiv:hep-th/0207130].
- [3] G. T. Horowitz and A. R. Steif, “Singular String Solutions With Nonsingular Initial Data,” Phys. Lett. B **258**, 91 (1991).
- [4] J. Khoury, B. A. Ovrut, N. Seiberg, P. J. Steinhardt and N. Turok, “From big crunch to big bang,” Phys. Rev. D **65**, 086007 (2002) [arXiv:hep-th/0108187].
- [5] N. Seiberg, “From big crunch to big bang - is it possible?,” arXiv:hep-th/0201039.
- [6] L. Cornalba and M. S. Costa, “A new cosmological scenario in string theory,” Phys. Rev. D **66**, 066001 (2002) [arXiv:hep-th/0203031].
- [7] N. A. Nekrasov, “Milne universe, tachyons, and quantum group,” arXiv:hep-th/0203112.
- [8] H. Liu, G. W. Moore and N. Seiberg, “The challenging cosmic singularity,” arXiv:gr-qc/0301001.
- [9] M. Berkooz, and B. Pioline, “Strings in an electric field, and the Milne universe,” JCAP **0311** (2003) 007 [arXiv:hep-th/0307280].
- [10] M. Berkooz, B. Pioline and M. Rozali, “Closed strings in Misner space: Cosmological production of winding strings,” JCAP **07** (2004) 003 [arXiv:hep-th/0405126].
- [11] M. Berkooz, Z. Komargodski, D. Reichmann and V. Shpitalnik, “Flow of geometries and instantons on the null orbifold,” arXiv:hep-th/0507067.
- [12] L. Cornalba and M. S. Costa, “Time-dependent orbifolds and string cosmology,” Fortsch. Phys. **52**, 145 (2004) [arXiv:hep-th/0310099].
- [13] B. Durin and B. Pioline, “Closed strings in Misner space: A toy model for a big bounce?,” arXiv:hep-th/0501145.

- [14] S. S. Gubser, “String production at the level of effective field theory,” *Phys. Rev. D* **69**, 123507 (2004) [arXiv:hep-th/0305099].
- [15] N. Turok and P. J. Steinhardt, “Beyond inflation: A cyclic universe scenario,” *Phys. Scripta* **T117**, 76 (2005) [arXiv:hep-th/0403020].
- [16] P. J. Steinhardt and N. Turok, “The cyclic model simplified,” *New Astron. Rev.* **49**, 43 (2005) [arXiv:astro-ph/0404480].
- [17] P. Horava and E. Witten, “Eleven-Dimensional Supergravity on a Manifold with Boundary,” *Nucl. Phys. B* **475**, 94 (1996) [arXiv:hep-th/9603142].
- [18] P. Horava and E. Witten, “Heterotic and type I string dynamics from eleven dimensions,” *Nucl. Phys. B* **460**, 506 (1996) [arXiv:hep-th/9510209].
- [19] N. Turok, M. Perry and P. J. Steinhardt, “M theory model of a big crunch / big bang transition,” *Phys. Rev. D* **70**, 106004 (2004) [Erratum-ibid. *D* **71**, 029901 (2005)] [arXiv:hep-th/0408083].
- [20] P. K. Townsend and M. N. R. Wohlfarth, “Accelerating cosmologies from compactification,” *Phys. Rev. Lett.* **91**, 061302 (2003) [arXiv:hep-th/0303097].
- [21] N. Ohta, “Accelerating cosmologies from S-branes,” *Phys. Rev. Lett.* **91**, 061303 (2003) [arXiv:hep-th/0303238].
- [22] N. Ohta, “A study of accelerating cosmologies from superstring / M theories,” *Prog. Theor. Phys.* **110**, 269 (2003) [arXiv:hep-th/0304172].
- [23] C. M. Chen, P. M. Ho, I. P. Neupane, N. Ohta and J. E. Wang, “Hyperbolic space cosmologies,” *JHEP* **0310**, 058 (2003) [arXiv:hep-th/0306291].
- [24] N. Ohta, “Accelerating cosmologies and inflation from M / superstring theories,” *Int. J. Mod. Phys. A* **20**, 1 (2005) [arXiv:hep-th/0411230].
- [25] R. H. Brandenberger and C. Vafa, “Superstrings In The Early Universe,” *Nucl. Phys. B* **316**, 391 (1989).
- [26] M. S. Costa, C. A. R. Herdeiro, J. Penedones and N. Sousa, “Hagedorn transition and chronology protection in string theory,” arXiv:hep-th/0504102.

- [27] A. J. Tolley, “String propagation through a big crunch / big bang transition,”  
arXiv:hep-th/0505158.
- [28] J. McGreevy and E. Silverstein, “The tachyon at the end of the universe,”  
arXiv:hep-th/0506130.
- [29] B. Craps, D. Kutasov and G. Rajesh, “String propagation in the presence of  
cosmological singularities,” *JHEP* **0206**, 053 (2002) [arXiv:hep-th/0205101].
- [30] E. Witten, “On string theory and black holes,” *Phys. Rev. D* **44**, 314 (1991).
- [31] D. Lust and S. Theisen, “Lectures in String Theory”, *Lecture Notes Phys.* 346  
(Springer-Verlag, Berlin, 1989).
- [32] J. Polchinski, “String Theory” (Cambridge Univer. Press, 1998), Vols. 1 and 2.